Chapter 6

Gravitation and Newton’s Synthesis
Units of Chapter 6

- Newton’s Law of Universal Gravitation
- Vector Form of Newton’s Law of Universal Gravitation
- Gravity Near the Earth’s Surface; Geophysical Applications
- Satellites and “Weightlessness”
- Kepler’s Laws and Newton’s Synthesis
- Gravitational Field

Opening Question

A space station revolves around the Earth as a satellite, 100 km above the Earth’s surface. What is the net force on an astronaut at rest inside the space station?

(a) Equal to her weight on Earth.
(b) A little less than her weight on Earth.
(c) Less than half her weight on Earth.
(d) Zero (she is weightless).
(e) Somewhat larger than her weight on earth.

In fact only about 3% less
If the force of gravity is being exerted on objects on Earth, what is the origin of that force?

Newton’s realization was that the force must come from the Earth.

He further realized that this force must be what keeps the Moon in its orbit.

The gravitational force on you is one-half of a third law pair: the Earth exerts a downward force on you, and you exert an upward force on the Earth.

When there is such a disparity in masses, the reaction force is undetectable, but for bodies more equal in mass it can be significant.
6-1 Newton’s Law of Universal Gravitation

Therefore, the gravitational force must be proportional to both masses.

By observing planetary orbits, Newton also concluded that the gravitational force must decrease as the inverse of the square of the distance between the masses.

In its final form, the law of universal gravitation reads:

\[ F = G \frac{m_1 m_2}{r^2}, \]

where

\[ G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2. \]

The magnitude of the gravitational constant \( G \) can be measured in the laboratory.

This is the Cavendish experiment.
6-1 Newton’s Law of Universal Gravitation

Schematic diagram of Cavendish’s apparatus. Two spheres are attached to a light horizontal rod, which is suspended at its center by a thin fiber. When a third sphere (A) is brought close to one of the suspended spheres, the gravitational force causes the latter to move, and this twists the fiber slightly. The tiny movement is magnified by the use of a narrow light beam directed at a mirror mounted on the fiber. The beam reflects onto a scale. Previous determination of how large a force will twist the fiber a given amount then allows the experimenter to determine the magnitude of the gravitational force between two objects.

Do not need to know details of this.

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6-1 Newton’s Law of Universal Gravitation

Example 6-1: Can you attract another person gravitationally?

A 50 kg person and a 70 kg person are sitting on a bench about 0.5 m apart. Estimate the magnitude of the gravitational force each exerts on the other.

\[
F = \frac{Gm_1m_2}{r^2}
\]

\[
= \frac{6.67 \times 10^{-11} \times 50 \times 70}{0.5^2} = 9.3 \times 10^{-7} \approx 10^{-6} \text{ N}
\]

This could not be detected without extremely sensitive instruments.
Example 6-2: Spacecraft at $2r_E$.

What is the force of gravity acting on a 2000 kg spacecraft when it orbits two Earth radii from the Earth’s center (that is, a distance $r_E = 6380$ km above the Earth’s surface)? The mass of the Earth is $m_E = 5.98 \times 10^{24}$ kg.

**Hard Way:**

$$ F = \frac{Gm_Em_s}{r_s^2} \quad (r_s = 2r_E) $$

$$ = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times 2000}{(2 \times 6.380 \times 10^8)^2} = 4900 \text{ N} $$

**Easy Way:**

Since $F \propto \frac{1}{r^2}$

If $r \rightarrow 2r$  $\Rightarrow$  $F \rightarrow \frac{1}{2^2} = \frac{1}{4}$

$\therefore$  $F = \frac{1}{4} mg = \frac{2000 \times 9.8}{4} = 4900 \text{ N}$
Example 6-3: Force on the Moon.

Find the net force on the Moon ($m_M = 7.35 \times 10^{22} \text{ kg}$) due to the gravitational attraction of both the Earth ($m_E = 5.98 \times 10^{24} \text{ kg}$) and the Sun ($m_S = 1.99 \times 10^{30} \text{ kg}$), assuming they are at right angles to each other.

\[
F = \sqrt{F_{ME}^2 + F_{MS}^2} = \sqrt{1.99 \times 10^{20} + 4.34 \times 10^{20}} = 4.77 \times 10^{20} \text{ N}
\]

\[
\theta = \tan^{-1}\left(\frac{F_{ME}}{F_{MS}}\right) = \tan^{-1}\left(\frac{1.99 \times 10^{20}}{4.34 \times 10^{30}}\right) = 24.6^\circ
\]
6-1 Newton’s Law of Universal Gravitation

Using calculus, you can show:

**Particle outside a thin spherical shell:**
gravitational force is the same as if all mass were at center of shell

**Particle inside a thin spherical shell:**
gravitational force is zero

Can model a sphere as a series of thin shells; outside any spherically symmetric mass, gravitational force acts as though all mass is at center of sphere.

6-2 Vector Form of Newton’s Universal Gravitation

In vector form,

\[ \vec{F}_{12} = -G \frac{m_1 m_2}{r_{21}^2} \hat{r}_{21}. \]

This figure gives the directions of the displacement and force vectors.
6-2 Vector Form of Newton’s Universal Gravitation

If there are many particles, the total force is the vector sum of the individual forces:

$$\vec{F}_i = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1n} = \sum_{i=2}^{n} \vec{F}_{1i}.$$ 

6-3 Gravity Near the Earth’s Surface; Geophysical Applications

Now we can relate the gravitational constant to the local acceleration of gravity. We know that, on the surface of the Earth:

$$mg = G \frac{mm_{E}}{r_{E}^2}.$$ 

Solving for $g$ gives:

$$g = G \frac{m_{E}}{r_{E}^2}.$$ 

Now, knowing $g$ and the radius of the Earth, the mass of the Earth can be calculated:

$$m_{E} = \frac{gr_{E}^2}{G} = \frac{9.80 \times (6.38 \times 10^6)^2}{6.67 \times 10^{-11}} = 5.98 \times 10^{24} \text{ kg}$$
Example 6-4: Gravity on Everest.

Estimate the effective value of $g$ on the top of Mt. Everest, 8850 m above sea level. That is, what is the acceleration due to gravity of objects allowed to fall freely at this altitude?

Mt. Everest: $h = 8850 \text{ m} = 8.85 \times 10^3 \text{ m}$ above sea level.

$r_E = 6380 \text{ km} = 6.380 \times 10^6 \text{ m}$

$r = r_E + h = 6.380 \times 10^6 + 8.85 \times 10^3 = 6.389 \times 10^6 \text{ m}$

From previous example: $g = \frac{G m_E}{r^2}$

\[
= \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.389 \times 10^6)^2} = 9.77 \text{ m/s}^2
\]
The acceleration due to gravity varies over the Earth’s surface due to altitude, local geology, and the shape of the Earth, which is not quite spherical.

### Example 6-5: Effect of Earth’s rotation on \( g \).

Assuming the Earth is a perfect sphere, determine how the Earth’s rotation affects the value of \( g \) at the equator compared to its value at the poles.
Example 6-5: Effect of Earth’s rotation on $g$

**At the pole** there are two forces acting on the person: $\vec{w}$ and $\vec{F}_g = mg$ where $\vec{w}$ is the reaction of the scales on the person.

As there is no rotation $w = mg$

**At the equator** there is a radial acceleration as the earth is rotating:

$$a_R = \frac{mv^2}{r_E}$$

The magnitude of $F_G = mg$ is unchanged

The scale pushes upwards with a force $w'$ which is equal to the force of the person on the scale.

From Newton’s 2nd law:

$$mg - w' = aR = \frac{mv^2}{r_E}$$

Solving for $w'$ gives:

$$w' = m\left(g - \frac{v^2}{r_E}\right)$$

---

Example 6-5: Effect of Earth’s rotation on $g$

Next calculate velocity:

$$v = \frac{\text{Circ}}{\text{time}} = \frac{2\pi r_E}{1 \text{ day}} = \frac{2\pi \times 6.38 \times 10^6 \text{ (m)}}{8.64 \times 10^5 \text{ (s)}} = 464.0 \text{ m/s}$$

As $w' = m\left(g - \frac{v^2}{r_E}\right)$ The effective weight $w' = mg'$

where $g'$ is the effective value of $g$

$$g' = \frac{w'}{m} = g - \frac{v^2}{r_E}$$

The change in $g$: $\Delta g = g - g' = \frac{v^2}{r_E}$

$$\Delta g = \frac{464}{6.38 \times 10^6} = 0.0337 \text{ m/s}^2 \text{ or about } 0.3\%$$

Note that as $\Delta g$ is positive $g'$ is negative

$g$ is slightly smaller at the equator compared to the poles.
6-4 Satellites and “Weightlessness”

Satellites are routinely put into orbit around the Earth. The tangential speed must be high enough so that the satellite does not return to Earth, but not so high that it escapes Earth’s gravity altogether.

The satellite is kept in orbit by its speed—it is continually falling, but the Earth curves from underneath it.
Example 6-6: Geosynchronous satellite.

(a) Relation between $v$ and $r \left[r = r_E + h; \text{where } h = \text{altitude}\right]$

For circular orbit: $F_c = \frac{mv^2}{r} = \frac{Gm_E}{r^2} \rightarrow \therefore v^2 = \frac{Gm_E}{r}$

For geosynchronous orbit:

\[
\begin{align*}
  v &= \frac{Circ}{time} = \frac{2\pi r}{t} = \frac{2\pi r}{86400} \quad \text{(1 day = 86400 s)}
\end{align*}
\]

Substitute into above equation and solve for $r$

\[
\begin{align*}
  r &= \frac{Gm_E}{v^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times 86400^2}{4\pi^2 r^2} = \frac{7.542 \times 10^{22}}{r^2}
\end{align*}
\]

\[
\begin{align*}
  \therefore r &= \sqrt[3]{7.542 \times 10^{22}} = 4.23 \times 10^7 \text{ m} = 42300 \text{ km}
\end{align*}
\]

\[
\begin{align*}
  \therefore h &= r - r_E = 42300 - 6380 = 35920 \text{ km} \approx 36000 \text{ km}
\end{align*}
\]

Example 6-6: Geosynchronous satellite.

(b) the satellite’s speed.

From previous slide:

\[
\begin{align*}
  v &= \sqrt{\frac{Gm_E}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{4.23 \times 10^7}} = 3070 \text{ m/s}
\end{align*}
\]

(c) Compare to the speed of a satellite orbiting 200 km above Earth’s surface.

From part (b) we see that $v \propto \sqrt{\frac{1}{r}}$

\[
\begin{align*}
  v' &= v \sqrt{\frac{r}{r'}} = 3070 \sqrt{\frac{42300}{6380 + 200}} = 7780 \text{ m/s}
\end{align*}
\]
Conceptual Example 6-7: Catching a satellite.

You are an astronaut in the space shuttle pursuing a satellite in need of repair. You find yourself in a circular orbit of the same radius as the satellite, but 30 km behind it. How will you catch up with it?

As \[ r = \frac{Gm_F}{v^2} \]  
If \( v \) increases – \( r \) must decrease 
or if \( v \) decreases – \( r \) must increase

You have to drop into a lower orbit to speed up; when you get ahead of the satellite you need to slow down and get back into the higher orbit.

6-4 Satellites and “Weightlessness”

Objects in orbit are said to experience weightlessness. They do have a gravitational force acting on them, though! The satellite and all its contents are in free fall, so there is no normal force. This is what leads to the experience of weightlessness.
6-4 Satellites and “Weightlessness”

More properly, this effect is called apparent weightlessness, because the gravitational force still exists. It can be experienced on Earth as well, but only briefly:

6-5 Kepler’s Laws and Newton’s Synthesis

Kepler’s laws describe planetary motion.
1. The orbit of each planet is an ellipse, with the Sun at one focus.

Kepler’s laws describe planetary motion.
1. The orbit of each planet is an ellipse, with the Sun at one focus.
2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

![Image of planetary orbits]

3. The square of a planet's orbital period is proportional to the cube of its mean distance from the Sun.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Mean Distance from Sun, ( s ) (10^8 km)</th>
<th>Period, ( T ) (Earth yr)</th>
<th>( s^3/T^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>57.9</td>
<td>0.241</td>
<td>3.34</td>
</tr>
<tr>
<td>Venus</td>
<td>108.2</td>
<td>0.615</td>
<td>3.35</td>
</tr>
<tr>
<td>Earth</td>
<td>149.6</td>
<td>1.0</td>
<td>3.35</td>
</tr>
<tr>
<td>Mars</td>
<td>227.9</td>
<td>1.88</td>
<td>3.35</td>
</tr>
<tr>
<td>Jupiter</td>
<td>778.3</td>
<td>11.86</td>
<td>3.35</td>
</tr>
<tr>
<td>Saturn</td>
<td>1427</td>
<td>29.5</td>
<td>3.34</td>
</tr>
<tr>
<td>Uranus</td>
<td>2870</td>
<td>84.0</td>
<td>3.35</td>
</tr>
<tr>
<td>Neptune</td>
<td>4497</td>
<td>165</td>
<td>3.34</td>
</tr>
<tr>
<td>Pluto</td>
<td>5900</td>
<td>248</td>
<td>3.34</td>
</tr>
</tbody>
</table>
Kepler’s laws can be derived from Newton’s laws. In particular, Kepler’s third law follows directly from the law of universal gravitation—equating the gravitational force with the centripetal force shows that, for any two planets (assuming circular orbits, and that the only gravitational influence is the Sun):

\[
\left( \frac{T_1}{T_2} \right)^2 = \left( \frac{r_1}{r_2} \right)^3.
\]

**Example 6-8: Where is Mars?**

Mars’ period (its “year”) was first noted by Kepler to be about 687 days (Earth-days), which is \((687 \text{ d}/365 \text{ d}) = 1.88 \text{ yr} \) (Earth years). Determine the mean distance of Mars from the Sun using the Earth as a reference. (\(r_{ES} = 1.50 \times 10^{11} \text{ m}\))

\[
\left( \frac{T_1}{T_2} \right)^2 = \left( \frac{r_1}{r_2} \right)^3
\]

\[
\therefore \frac{r_{MS}}{r_{ES}} = \left( \frac{T_M}{T_E} \right)^{2/3} = \left( \frac{1.88}{1} \right)^{2/3} = 1.52
\]

\[
\therefore r_{MS} = 1.52 \times 1.50 \times 10^{11} = 2.28 \times 10^{11} \text{ m}
\]
6-5 Kepler’s Laws and Newton's Synthesis

Example 6-9: The Sun’s mass determined.

Determine the mass of the Sun given the Earth’s distance from the Sun as $r_{ES} = 1.5 \times 10^{11}$ m.

(Note: This an easier approach than Giancoli without having to prove Kepler’s 3rd law.)

\[
F_C = \frac{m_E v_E^2}{r_{ES}} = \frac{Gm_E m_S}{(r_{ES})^2}
\]

and

\[
v_E = \frac{2\pi r_{ES}}{t} = \frac{2\pi \times 1.5 \times 10^{11}}{365 \times 24 \times 60 \times 60} = 3.00 \times 10^4 \text{ m/s}
\]

\[. \]

\[
m_S = \frac{r_{ES} v_E^2}{G} = \frac{1.5 \times 10^{11} \times (3.00 \times 10^4)^2}{6.67 \times 10^{-11}}
\]

\[= 2.0 \times 10^{30} \text{ kg}
\]

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6-5 Kepler’s Laws and Newton's Synthesis

Irregularities in planetary motion led to the discovery of Neptune, and irregularities in stellar motion have led to the discovery of many planets outside our solar system.
6-6 Gravitational Field

The gravitational field is the gravitational force per unit mass:

\[ \vec{g} = \frac{\vec{F}}{m}. \]

The gravitational field due to a single mass \( M \) is given by:

\[ \vec{g} = -\frac{GM}{r^2} \hat{r}. \]

Summary of Chapter 6

• Newton’s law of universal gravitation:

\[ F = G \frac{m_1 m_2}{r^2} \]

• Total force is the vector sum of individual forces.

• Satellites are able to stay in Earth orbit because of their large tangential speed.

• Newton’s laws provide a theoretical base for Kepler’s laws.