Chapter 13
Fluids
**Units of Chapter 13**

- Phases of Matter
- Density and Specific Gravity
- Pressure in Fluids
- Atmospheric Pressure and Gauge Pressure
- Pascal’s Principle
- Measurement of Pressure; Gauges and the Barometer
- Buoyancy and Archimedes’ Principle

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**Units of Chapter 13**

- Fluids in Motion; Flow Rate and the Equation of Continuity
- Bernoulli’s Equation
- Applications of Bernoulli’s Principle: Torricelli, Airplanes, Baseballs, TIA
- Viscosity
- Flow in Tubes: Poiseuille’s Equation, Blood Flow
13-1 Phases of Matter
The three common phases of matter are solid, liquid, and gas.
A solid has a definite shape and size.
A liquid has a fixed volume but can be any shape.
A gas can be any shape and also can be easily compressed.
Liquids and gases both flow, and are called fluids.

13-2 Density and Specific Gravity
The density $\rho$ of a substance is its mass per unit volume:
$$\rho = \frac{m}{V}.$$  

The SI unit for density is kg/m$^3$. Density is also sometimes given in g/cm$^3$; to convert g/cm$^3$ to kg/m$^3$, multiply by 1000.

Water at 4°C has a density of 1 g/cm$^3 = 1000$ kg/m$^3$.

The specific gravity of a substance is the ratio of its density to that of water.
13-2 Density and Specific Gravity

Example 13-1: Mass, given volume and density.

What is the mass of a solid iron wrecking ball of radius 18 cm?

\[ \rho_{\text{iron}} = 7.8 \times 10^3 \text{ kg/m}^3 \]
\[ r = 18 \text{ cm} = 0.18 \text{ m} \]
\[ V = \frac{4}{3} \pi r^3 = \frac{4\pi(0.18)^3}{3} = 0.024 \text{ m}^3 \]
\[ \rho = \frac{m}{V} \rightarrow m = \rho V = 7.8 \times 10^3 \times 0.024 = 190 \text{ kg} \]
13-3 Pressure in Fluids

Pressure is defined as the force per unit area.

\[
\text{pressure } = P = \frac{F}{A}.
\]

Pressure is a scalar; the units of pressure in the SI system is the pascal:

\[
1 \text{ Pa} = 1 \text{ N/m}^2.
\]

Example 13-2: Calculating pressure.

The two feet of a 60 kg person cover an area of 500 cm².

(a) Determine the pressure exerted by the two feet on the ground.

(b) If the person stands on one foot, what will the pressure be under that foot?

(a) \( A = 500 \text{ cm}^2 = 500 \times 10^{-4} = 5 \times 10^{-2} \text{ m}^2 \)

\( F = mg = 60 \times 9.8 = 588 \text{ N} \)

\( P = \frac{F}{A} = \frac{588}{(5 \times 10^{-2})} = 1.2 \times 10^4 \text{ Pa} \)

(b) Area halved \( \rightarrow \) Pressure doubled

\( P = 2.4 \times 10^4 \text{ Pa} \)
Pressure is a scalar (it has a magnitude but no direction) and is exerted equally in all directions.

Example:

A living room has floor dimensions of 3.5 m x 4.2 m and height of 2.4 m.

(a) What does the air in the room weigh?

(b) What force does the atmosphere exert on the floor of the room?

(a) What does the air in the room weigh?

Volume of room: \( V = 3.5 \times 4.2 \times 2.4 = 35.28 \text{ m}^3 \)
Density of air: \( \rho_{\text{air}} = 1.29 \text{ kg/m}^3 \)
Mass of air: \( \rho = \frac{m}{V} \rightarrow m = \rho V = 1.29 \times 35.28 = 45.51 \text{ kg} \)
Weight of air: \( W = mg = 45.51 \times 9.8 = 446 \text{ N} \)

(b) What force does the atmosphere exert on the floor of the room?

Atmospheric pressure: \( P_{\text{atm}} = 1.013 \times 10^5 \text{ Pa} \)
Floor area: \( A = 3.5 \times 4.2 = 14.7 \text{ m}^2 \)
Force on floor: \( P = \frac{F}{A} \)\)
\( \rightarrow F = PA = 1.013 \times 10^5 \times 14.7 = 1.5 \times 10^6 \text{ N} \)

Much greater than the weight of air in the room. \( \rightarrow \) It is the weight of a column of air (same area as floor) extending from ground level to the top of the atmosphere.
13-3 Pressure in Fluids

Pressure is the same in every direction in a static fluid at a given depth; if it were not, the fluid would flow.

For a fluid at rest, there is also no component of force parallel to any solid surface—once again, if there were, the fluid would flow.

\[ F_{||} = 0 \]

So force is perpendicular to the surface.
13-3 Pressure in Fluids

The pressure at a depth $h$ below the surface of the liquid is due to the weight of the liquid above it. We can quickly calculate:

$$P = \frac{F}{A} = \frac{\rho Ahg}{A}$$

$$P = \rho gh.$$  

This relation is valid for any liquid whose density does not change with depth.

13-3 Pressure in Fluids

If there is external pressure in addition to the weight of the fluid itself, or if the density of the fluid is not constant, we calculate the pressure at a height $y$ in the fluid; the negative sign indicates that the pressure decreases with height (increases with depth):

$$\frac{dP}{dy} = -\rho g.$$
13-3 Pressure in Fluids

We then integrate to find the pressure:

\[ \int_{P_1}^{P_2} dP = -\int_{y_1}^{y_2} \rho g \, dy \]

\[ P_2 - P_1 = -\int_{y_1}^{y_2} \rho g \, dy. \]

Pressure at a depth \( h = (y_2 - y_1) \) in a liquid of density \( \rho \) is

\[ P = P_0 + \rho gh \]

where \( P_0 \) is the external pressure at the liquid’s top surface.

Example 13-3: Pressure at a faucet.

The surface of the water in a storage tank is 30 m above a water faucet in the kitchen of a house. Calculate the difference in water pressure between the faucet and the surface of the water in the tank.

\[ P = P_0 + \rho gh \]

\[ \Delta P = P - P_0 = \rho gh \]

\[ = 1000 \times 9.8 \times 30 = 2.9 \times 10^5 \text{ Pa} \]

Calculate the force due to water pressure exerted on a 1.0 m x 3.0 m aquarium viewing window whose top edge is 1.0 m below the water surface.

\[ F = 3 \rho g \int_0^2 y \, dy = 3 \rho g \left[ \frac{y^2}{2} \right]_0^2 \]
\[ = 3 \rho g \left( \frac{4}{2} - \frac{1}{2} \right) = \frac{9 \rho g}{2} \]
\[ = \frac{9 \times 1000 \times 9.8}{2} = 44000 \text{ N} \]
13-3 Pressure in Fluids

Example 13-5: Elevation effect on atmospheric pressure.

(a) Determine the variation in pressure in the Earth’s atmosphere as a function of height $y$ above sea level, assuming $g$ is constant and that the density of the air is proportional to the pressure. (This last assumption is not terribly accurate, in part because temperature and other weather effects are important.)

(b) At what elevation is the air pressure equal to half the pressure at sea level?

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13-3 Pressure in Fluids

Example 13-5: Elevation effect on atmospheric pressure.

(a) As $\rho \propto P \rightarrow \frac{\rho}{\rho_0} = \frac{P}{P_0}$

$P_0 = 1.013 \times 10^5$ Pa = Atmospheric pressure at sea level

$\rho_0 = 1.29$ kg/m$^3$ = Density at sea level

$dP = -\rho g \, dy \rightarrow \frac{dP}{dy} = -\rho g = -P\left(\frac{P_0}{P_0}\right)g$ (-ve as pressure $\downarrow$ with height)

$\int_0^P \frac{dP}{P} = -\frac{\rho_0}{P_0} g \int_0^y \, dy = \ln \frac{P}{P_0} - g'y$

since $\ln P - \ln P_0 = \ln(P/P_0)$, then $P = P_0 e^{-(\rho_0 g / P_0) y}$

Pressure decreases approximately exponentially with height.

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Example 13-5: Elevation effect on atmospheric pressure.

(b) At what elevation is the air pressure equal to half the pressure at sea level?

\[ P_0 = 1.013 \times 10^5 \text{ Pa} = \text{Atmospheric pressure at sea level} \]
\[ \rho_0 = 1.29 \text{ kg/m}^3 = \text{Density at sea level} \]
\[ P = P_0 e^{-\left(\frac{\rho_g}{\rho_0}\right)y} \]

For \( P = P_0 / 2 \)

\[ \frac{P}{P_0} = e^{-\left(\frac{y}{y_0}\right)} = e^{-\left(\frac{1.29 \times 10^3}{1.013 \times 10^5}\right)} \]
\[ e^{1.248 \times 10^{-4} y} = 2 \]
\[ 1.248 \times 10^{-4} y = \ln 2 = 0.693 \]
\[ y = \frac{0.693}{1.248 \times 10^{-4}} = 5550 \text{ m} \]

13-4 Atmospheric Pressure and Gauge Pressure

At sea level the atmospheric pressure is about 1.013 x 10^5 Pa ; this is called 1 atmosphere (atm).

Another unit of pressure is the bar:

1 bar = 1.00 x 10^5 Pa.

Standard atmospheric pressure is just over 1 bar.

This pressure does not crush us, as our cells maintain an internal pressure that balances it.

Meteorologists use the milli bar = 100 Pa on weather maps.

(Standard atmospheric pressure = 1013 milli bar)
Conceptual Example 13-6: Finger holds water in a straw.

You insert a straw of length \( l \) into a tall glass of water. You place your finger over the top of the straw, capturing some air above the water but preventing any additional air from getting in or out, and then you lift the straw from the water. You find that the straw retains most of the water. Does the air in the space between your finger and the top of the water have a pressure \( P \) that is greater than, equal to, or less than the atmospheric pressure \( P_0 \) outside the straw?

There must be a net upward force on the water in the straw to keep it from falling out; therefore the pressure in the space above the water must be less than atmospheric pressure.

Most pressure gauges measure the pressure above the atmospheric pressure—this is called the gauge pressure.

The absolute pressure is the sum of the atmospheric pressure and the gauge pressure.

\[
P = P_0 + P_G.
\]

Determine the gauge pressure at a house which is situated 100 m below the level of the storage dam? \( (9.8 \times 10^5 \text{ Pa}) \)

(Use \( P = \rho gh \))
More Sample Problems

1. In water the pressure increases with depth according to \( P = \rho gh \). At what depth below the surface would the pressure on a scuba diver be 1 atmosphere above the pressure at the surface?

Solution

If \( P = P_{\text{atm}} = 1.013 \times 10^5 \text{ Pa} \), \( \rho_{\text{sea water}} = 1025 \text{ kg/m}^3 \), and \( g = 9.8 \text{ m/s}^2 \), then the depth \( h \) is given by:

\[
h = \frac{P}{\rho g} = \frac{1.013 \times 10^5}{(1025 \times 9.8)} = 10.08 \text{ m}.
\]

When you are diving (in the sea) the pressure on your body increases by about 1 atm for every 10 m under water.

When scuba diving, the demand valve supplies air at water pressure so that the pressure in the chest cavity is almost the same as the water pressure so that your chest is not crushed.

Note that the solubility of nitrogen in blood plasma increases as the pressure of the air breathed by a scuba diver increases with depth.

When diving at greater depths it is essential to ascend slowly, pausing at regular intervals to avoid the bends (gas bubbles forming in blood vessels and tissue).

If the diver surfaces too rapidly, nitrogen dissolved in the blood will come out of solution and form bubbles in the blood vessels. This causes occlusion of the capillaries and causes the bends.
This is life threatening situation and is normally treated in a hyperbaric chamber.

The air pressure in the chamber is increased to the equivalent of the pressure at the maximum depth that the diver attained and slowly decreased to atmospheric pressure so that the nitrogen can be eliminated through the lungs.

For very deep dives this problem can largely be avoided by breathing a helium – oxygen mixture as helium is almost insoluble in blood plasma.

2. The Russian submarine “Kursk” sank in a terrible tragedy in 2001. After taking on water, the submarine sank to the continental shelf only 100 m below the surface.

(a) What was the absolute pressure at this depth?
(b) How many times greater is this than atmospheric pressure?

**Solution:**

Pressure at surface: \( P_0 = P_{\text{atm}} = 1.013 \times 10^5 \text{ Pa} \)
Density of sea water: \( \rho = 1025 \text{ kg/m}^3 \)
Depth of the water: \( d = 100 \text{ m} \)

\[
P = P_0 + \rho gh = 1.013 \times 10^5 + 1025 \times 9.8 \times 100 = 1.106 \times 10^6 \text{ Pa}
\]

\[
= 1106 \text{ kPa}
\]

Since atmospheric pressure \( P_{\text{atm}} = 1.013 \times 10^5 \text{ Pa} = 101.3 \text{ kPa} \)

This is \(\frac{1106}{101.3} = 10.9 \times \text{atmospheric pressure}\)
3. The water in a marine aquarium is 5 m deep.
   (a) What is the pressure at the surface?
   (b) What is the water pressure at the bottom?

**Solution:**

(a) Pressure at the surface: $P_{\text{atm}} = 1.013 \times 10^5 \text{ Pa} = 101.3 \text{ kPa}$

(b) Pressure at bottom:

$$P = P_0 + \rho gh = 1.013 \times 10^5 + 1025 \times 9.8 \times 5 = 1.515 \times 10^5 \text{ Pa}$$

$$= 151.5 \text{ kPa}$$

4. Vacuum cleaner salespeople use the “bowling ball” demonstration to “prove” how strong “suction” is. In fact there is no “suction” force rather atmospheric pressure is pushing on the ball from all sides.

In this problem let’s imagine we have a flat square metal plate 100 mm x 100 mm in size. On one side is vacuum ($P = 0$), on the other is atmospheric pressure of 1.0 atm = $1.013 \times 10^5 \text{ Pa}$.

What is the mass of the plate that can be supported by “suction” if the vacuum seal is perfect?
Solution:

For the maximum supportable mass the weight equals the pressure force:

\[ F_p = PA = F_w = mg \]
\[ \therefore \ mg = PA \]

& Area of metal plate: \( A = 10^{-1} \times 10^{-1} = 10^{-2} \text{ m}^2 \)
\[ \therefore \ m = \frac{PA}{g} = (1.013 \times 10^5 \times 10^{-2})/9.8 = 103.4 \text{ kg} \]

So you can see that air pressure can hold 103 kg if the vacuum on the other side is perfect.

5. Water has a density of 998 kg/m³ (at 20°C). What is the density of water in g/cm³?
   (0.998 g/cm³)

6. The density of water at 4°C is 1000 kg/m³. What is the mass of one litre of pure water? What are the dimensions in mm of a cube which has a volume of 1 L?
   (1 kg, 100 mm).

7. Air pressure at sea-level is 101.3 kPa. The surface area of an adult human is ~ 2.5 m². What force is the air exerting on us at all times? Why don’t we feel it? What happens to us in a vacuum?
   (253 kN)
8. The gravity fed water reservoir for a city is located 40 m above the city. Calculate the water pressure at homes in the city in units of \( \text{N/m}^2 \), \( \text{atm} \) and \( \text{mmHg} \).

(Use \( \rho_{\text{water}} = 1000 \text{ kg/m}^3 \))

\[
(3.92 \times 10^5 \text{ N/m}^2, \ 3.9 \text{ atm}, \ 2941 \text{ mmHg})
\]

Note:

1. \( \text{N/m}^2 = \text{Pa} \)
2. \( 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} = 760 \text{ mmHg} \)

\[
\frac{1.013 \times 10^5 (\text{Pa})}{3.92 \times 10^5 (\text{Pa})} = \frac{760 (\text{mmHg})}{P (\text{mmHg})} \quad \rightarrow \quad P = \frac{3.92 \times 10^5 \times 760}{1.013 \times 10^5} = 2941 \text{ mmHg}
\]

3. Using ratios is the easiest way to convert between \( \text{Pa} \) and \( \text{mmHg} \).

\[
P_{\text{atm}} (\text{Pa}) = P_{\text{atm}} (\text{mmHg}) \quad \rightarrow \quad P (\text{mmHg}) = \frac{P_{\text{atm}} (\text{mmHg}) \cdot P (\text{Pa})}{P_{\text{atm}} (\text{Pa})} = \frac{760}{1.013 \times 10^5} \cdot P (\text{Pa})
\]

\[
= 7.503 \times 10^{-3} \cdot P (\text{Pa})
\]

---

13-5 Pascal’s Principle

**If an external pressure is applied to a confined fluid, the pressure at every point within the fluid increases by that amount.**

This principle is used, for example, in hydraulic lifts and hydraulic brakes.
Hydraulic Lift

Two cylinders connected by a tube are filled with an incompressible fluid (eg. oil). The fluid is confined in the system by two pistons, as shown at right.

When force $F_1$ is exerted on the small piston, fluid flows from the small cylinder to the large one, moving the large piston up and exerting an upward force $F_2$. Pascal’s principle states that the pressure created by exerting the force $F_1$ on the small piston, $P_1 = \frac{F_1}{A_1}$, is transmitted undiminished to all parts of the fluid.

Therefore, $P_1 = P_2$ and $\frac{F_1}{A_1} = \frac{F_2}{A_2}$

$F_2 = \frac{A_2}{A_1} F_1$

Hydraulic Lift (Cont)

The force is magnified by the ratio of the areas. Often a lever is used to magnify the force $F_1$, even further so that very small forces can lift heavy objects, eg. hydraulic car jack.

Problem:

In a hydraulic press the smaller piston has a diameter of 40 mm and the larger piston has a diameter of 500 mm. What force must be applied to the smaller cylinder to lift a mass of 2000 kg? If it is required to lift the 2000 kg mass 100 mm, how far must the small piston move?
Solution:

\[ \frac{F_1}{A_1} = \frac{F_2}{A_2} \rightarrow F_2 = \frac{A_1F_1}{A_2} = \frac{kd_1^2F_1}{ kd_2^2} = \frac{d_1^2F_1}{d_2^2} = \frac{0.04^2 \times 19600}{0.5^2} = 125 \text{ N} \]

The volume must be the same on both sides.

\[ V_1 = V_2 \rightarrow \frac{\pi d_1^2 h_1}{4} = \frac{\pi d_2^2 h_2}{4} \]

\[ \therefore d_1^2 h_1 = d_2^2 h_2 \rightarrow h_1 = \frac{d_2^2 h_2}{d_1^2} = \frac{0.5^2 \times 0.1}{0.04^2} = 15.625 \text{ m} \]

Note that this is impractical. In practice a lever would be used to apply force \( F_1 \), and valves used so that the smaller cylinder would refill after each pump of the handle.

If each time you pumped the handle, piston 1 moved 50 mm, how many times would you have to pump the handle to lift the 2000 kg mass by 100 mm?

\[ d_1 = 40 \text{ mm} = 0.04 \text{ m} \]
\[ d_2 = 500 \text{ mm} = 0.5 \text{ m} \]

Since \( A = \pi d^2/4 \rightarrow A = kd^2 \)

\[ A_1 = kd_1^2 \]
\[ A_2 = kd_2^2 \]

\[ F_2 = m_2 g = 2000 \times 9.8 = 19600 \text{ N} \]

\[ h_2 = 100 \text{ mm} = 0.1 \text{ m} \]

Solution:

Each pump of the handle pushes volume

\[ V = \pi d_1^2 h / 4 = \left( \pi (0.04)^2 \times 0.05 \right) / 4 = 2\pi \times 10^{-3} \text{ m}^3 \text{ into piston 2} \]

The volume moved by piston 2

\[ V_2 = \pi d_2^2 h / 4 = \left( \pi (0.5)^2 \times 0.1 \right) / 4 = 6.25\pi \times 10^{-3} \text{ m}^3 \]

\[ \therefore \text{ Number of handle pumps} = \left( \pi \times 6.25 \times 10^{-3} \right) / \left( \pi \times 2 \times 10^{-5} \right) = 313 \]
There are a number of different types of pressure gauges. This one is an open-tube manometer. The pressure in the open end is atmospheric pressure; the pressure being measured will cause the fluid to rise until the pressures on both sides at the same height are equal.

Here are two more devices for measuring pressure: the aneroid gauge and the tire pressure gauge.
### 13-6 Measurement of Pressure; Gauges and the Barometer

Pressure is measured in a variety of different units. This table gives the conversion factors.

<table>
<thead>
<tr>
<th>TABLE 13–2 Conversion Factors Between Different Units of Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>In Terms of 1 Pa = 1 N/m²</strong></td>
</tr>
<tr>
<td>1 atm = $1.013 \times 10^5$ N/m²</td>
</tr>
<tr>
<td>1 bar = $1.000 \times 10^5$ N/m²</td>
</tr>
<tr>
<td>1 dyne/cm² = 0.1 N/m²</td>
</tr>
<tr>
<td>1 lb/in.² = $6.90 \times 10^3$ N/m²</td>
</tr>
<tr>
<td>1 lb/ft² = $47.9$ N/m²</td>
</tr>
<tr>
<td>1 cm-Hg = $1.33 \times 10^3$ N/m²</td>
</tr>
<tr>
<td>1 mm-Hg = $133$ N/m²</td>
</tr>
<tr>
<td>1 torr = 133 N/m²</td>
</tr>
<tr>
<td>1 mm-H₂O (4°C) = $9.80$ N/m²</td>
</tr>
</tbody>
</table>

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### 13-6 Measurement of Pressure; Gauges and the Barometer

This is a mercury barometer, developed by Torricelli to measure atmospheric pressure. The height of the column of mercury is such that the pressure in the tube at the surface level is 1 atm.

Therefore, pressure is often quoted in millimeters (or inches) of mercury.

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Any liquid can serve in a Torricelli-style barometer, but the most dense ones are the most convenient. This barometer uses water.

A water barometer: a full tube of water is inserted into a tub of water, keeping the tube’s spigot at the top closed. When the bottom end of the tube is unplugged, some water flows out of the tube into the tub, leaving a vacuum between the water’s upper surface and the spigot. Why? Because air pressure can not support a column of water more than 10 m high.

13-6 Measurement of Pressure; Gauges and the Barometer

Conceptual Example 13-7: Suction.

A student suggests suction-cup shoes for Space Shuttle astronauts working on the exterior of a spacecraft. Having just studied this Chapter, you gently remind him of the fallacy of this plan. What is it?

Suction cups work because of air pressure, and there isn’t any air where the shuttle orbits.
The pressure difference between the container and the atmospheric pressure is shown by \textit{difference} in the height of the \textit{working fluid} between the two sides of the manometer.

### Problems:

1. In the previous diagram an oil ($\rho = 1900 \, \text{kg/m}^3$) is used as working fluid. The difference in height $h$ of the working fluid is 100 mm. What is the pressure in the container? What is the gauge pressure?

### Solution:

The pressure in the container is given by:  
\[ P_{\text{container}} = P_0 + \rho_{\text{wf}} gh \]

As atmospheric pressure is 101.3 kPa and the density of oil is $1.9 \times 10^3 \, \text{kg/m}^3$.

\[ P = 1.013 \times 10^5 + 1.9 \times 10^3 \times 9.8 \times 0.1 = 1.032 \times 10^5 \, \text{Pa} \]

\[ = 103.2 \, \text{kPa} \]

\[ P_{\text{gauge}} = P - P_0 = 103.2 - 101.3 = 1.9 \, \text{kPa} \]
2. Compare the heights of a water column and mercury column when used in a barometer.

Solution:

\[ \rho_{\text{water}} = 1000 \text{ kg/m}^3 \]
\[ \rho_{\text{Hg}} = 13600 \text{ kg/m}^3 \]

Water column will be \( \frac{13600}{1000} = 13.6 \text{ times} \) the height of mercury column.

13-7 Buoyancy and Archimedes’ Principle

This is an object submerged in a fluid. There is a net force on the object because the pressures at the top and bottom of it are different.

The buoyant force is found to be the upward force on the same volume of water:

\[ F_B = F_2 - F_1 = \rho_F g A (h_2 - h_1) \]
\[ = \rho_F g A \Delta h \]
\[ = \rho_F V g \]
\[ = m_F g. \]
Archimedes’ principle:

The buoyant force on an object immersed in a fluid is equal to the weight of the fluid displaced by that object.

Conceptual Example 13-8: Two pails of water.

Consider two identical pails of water filled to the brim. One pail contains only water, the other has a piece of wood floating in it. Which pail has the greater weight?

Both weigh the same; if both pails were full to the brim before the wood was put in, some water will have spilled out.
Example 13-9: Recovering a submerged statue.

A 70 kg ancient statue lies at the bottom of the sea. Its volume is $3.0 \times 10^4 \text{ cm}^3$. How much force is needed to lift it?

The buoyancy force is equal to the weight of the displaced water

$$V = 3.0 \times 10^4 \text{ cm}^3 = 3 \times 10^{-2} \text{ m}^3$$

$$F_B = m_{\text{water}}g = \rho_{\text{water}}Vg$$

$$= 1000 \times 3 \times 10^{-2} \times 9.8 = 294 \text{ N}$$

$$F = mg - F_B = 70 \times 9.8 - 294 = 390 \text{ N}$$

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Example 13-10: Archimedes: Is the crown gold?

When a crown of mass 14.7 kg is submerged in water, an accurate scale reads only 13.4 kg. Is the crown made of gold?
Example 13-10: Archimedes: Is the crown gold?

Buoyancy Force $F_B = \text{weight of displaced water} = w - w' = (14.7 - 13.4)g = 1.3g = N$

Mass of displaced water $= 1.3 \text{ kg}$

$\rho_{\text{H}_2\text{O}} = \frac{m}{V} \rightarrow V = \frac{m}{\rho} = \frac{1.3}{1000} = 1.3 \times 10^{-3} \text{ m}^3$

$\rho_{\text{crown}} = \frac{m_{\text{crown}}}{V} = \frac{14.7}{1.3 \times 10^{-3}} = 11.3 \times 10^3 \text{ kg/m}^3$

which suggests that the crown is made of lead.

The density of gold is $19.3 \times 10^3 \text{ kg/m}^3$

A widely known anecdote about Archimedes tells how he invented a method for determining the volume of an object with an irregular shape. According to Vitruvius, a votive crown for a temple had been made for King Hiero II of Syracuse (308 - 215 BC), who had supplied the pure gold to be used, and Archimedes was asked to determine whether some silver had been substituted by the dishonest goldsmith.

Archimedes had to solve the problem without damaging the crown, so he could not melt it down into a regularly shaped body in order to calculate its density. He deduced that if an object was immersed in water, it would displace a volume of water equal to the volume of the object.

From mass and volume the density of the crown could be determined. The density would be lower than that of gold if cheaper and less dense metals had been added. The test was conducted successfully, proving that silver had indeed been mixed in (Read goldsmith was executed).
This story does not appear in the known works of Archimedes and the practicality of the method has been called into question, due to the extreme accuracy with which one would have to measure the water displacement.

Archimedes may have used a method based on his principle (described in his treatise ‘On Floating Bodies’. Using this principle, it would have been possible to compare the density of the golden crown to that of solid gold by balancing the crown on a scale with a gold reference sample, then immersing the apparatus in water. The difference in density between the two samples would cause the scale to tip accordingly. Galileo considered it "probable that this method is the same that Archimedes followed, since, besides being very accurate, it is based on demonstrations found by Archimedes himself."

As the crown is not pure gold (has a larger volume than the same mass of gold) it will displace a larger volume of water than the gold and experience a greater buoyancy force. ie. It will appear lighter than an equal mass of pure gold when immersed in water.
13-7 Buoyancy and Archimedes’ Principle

If an object’s density is less than that of water, there will be an upward net force on it, and it will rise until it is partially out of the water.

For a floating object, the fraction that is submerged is given by the ratio of the object’s density to that of the fluid.

\[
\frac{V_{\text{displ}}}{V_O} = \frac{\rho_O}{\rho_F}
\]

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13-7 Buoyancy and Archimedes’ Principle

Example 13-11: Hydrometer calibration.

A hydrometer is a simple instrument used to measure the specific gravity of a liquid by indicating how deeply the instrument sinks in the liquid. This hydrometer consists of a glass tube, weighted at the bottom, which is 25.0 cm long and 2.00 cm² in cross-sectional area, and has a mass of 45.0 g. How far from the end should the 1.000 mark be placed?

Example 13-11: Hydrometer calibration.

Average density of hydrometer:

$$\rho_{hi} = \frac{m}{V} = \frac{0.045}{0.25 \times 2.0 \times 10^{-3}} = 900 \text{ kg/m}^3$$

Hydrometer will float with 900/1000 = 0.90 under surface

$$x = 0.90 \times 25.0 = 22.5 \text{ cm from bottom}$$
Example 13-12: Helium balloon.

What volume \( V \) of helium is needed if a balloon is to lift a load of 180 kg (including the weight of the empty balloon)?

\[
\rho_{\text{He}} = 0.179 \text{ kg/m}^3 \quad \rho_{\text{air}} = 1.29 \text{ kg/m}^3
\]

\[
F_B = \text{weight of displaced air} = m_{\text{air}}g = \rho_{\text{air}}Vg
\]

\[
F_B = m_{\text{He}}g + m_{\text{load}}g = \rho_{\text{He}}Vg + m_{\text{load}}g
\]

\[
\rho_{\text{air}}V = \rho_{\text{He}}V + m_{\text{load}}
\]

\[
V = \frac{m_{\text{load}}}{\rho_{\text{air}} - \rho_{\text{He}}} = \frac{180}{1.29 - 0.179} = 160 \text{ m}^3
\]
More Problems:

14. What is the buoyancy force due to the earth’s atmosphere on an average human (height 1.83 m, weight = 70 kg, density = 1030 kg/m³).

Solution:

\[ \rho = \frac{m}{V} \rightarrow V = \frac{m}{\rho} = \frac{70}{1030} = 0.068 \text{ m}^3 \]

Weight of air displaced (\( \rho_{\text{air}} = 1.29 \text{ kg/m}^3 \))

\[ W = mg = \rho V g = 1.29 \times 0.068 \times 9.8 = 0.860 \text{ N} \]

\( \therefore \) The buoyancy force due to the air is only 0.860 N.

Since \( W = mg \rightarrow m = \frac{W}{g} = \frac{0.860}{9.8} = 0.088 \text{ kg} \) which is only an apparent reduction in mass of

\( (0.088/70) \times 100 = 0.1\% \) of our total mass.

If we were immersed in a more dense fluid e.g. water: 
\( \rho = 1000 \text{ kg/m}^3 \) the buoyancy force would be greater and we would feel lighter.

Weight of displaced water = \( \rho_{\text{water}} V g = 1000 \times 0.068 \times 9.8 \)

\[ = 666 \text{ N} \]

Total weight of the person is: \( W = mg = 70 \times 9.8 = 686 \text{ N} \), this is a reduction in apparent weight of \( (686 - 666)/666 = 0.003 \). i.e. When immersed in water the apparent weight of the person would have only 0.3% of their weight in air.
2. There is a common saying “just the tip of the iceberg”, because icebergs are \(\approx 90\%\) underwater. Given the densities of ice and sea-water we can calculate the volume of an iceberg that lies below the water line.

Density of ice: \(\rho_{\text{ice}} = 917 \text{ kg/m}^3\)
Density sea water: \(\rho_{\text{sw}} = 1025 \text{ kg/m}^3\)

Solution:

The solution to this problem is (slightly) difficult.

First assume the iceberg has volume \(V_B\). Using the density of ice calculate the mass of ice:

\[
\rho_B = \frac{m_B}{V_B} \rightarrow m_B = \rho_B V_B = 917 V_B
\]

As the ice floats, we know the weight of the water displaced must be the same as the weight of the entire iceberg.

Weight of water displaced: \(W_W = W_B \rightarrow m_{Wg} = m_B \rightarrow m_W = m_B\)

Since \(m = \rho V \rightarrow m_W = \rho_W V_W = m_B = \rho_B V_B \rightarrow \rho_W V_W = \rho_B V_B\)

\[
\rightarrow V_W/V_B = \rho_B/\rho_W = 917/1025 = 0.895
\]

This shows us that \(\approx 90\%\) of the volume of the iceberg is underwater.
If the flow of a fluid is smooth, it is called streamline or laminar flow (a).

Above a certain speed, the flow becomes turbulent (b). Turbulent flow has eddies; the viscosity of the fluid is much greater when eddies are present.

Streamlines

A streamline is a path traced out by a tiny fluid element when fluid flows.

The local fluid velocity is always at a tangent to the streamline.

Streamlines cannot cross. If they did the fluid would be going in two directions at once.
13-8 Fluids in Motion; Flow Rate and the Equation of Continuity

We will deal first with laminar flow.

The mass flow rate is the mass that passes a given point per unit time. The flow rates at any two points must be equal, as long as no fluid is being added or taken away.

This gives us the equation of continuity:

\[
\frac{\Delta m_1}{\Delta t} = \frac{\Delta m_2}{\Delta t},
\]

Since

\[ \rho_1 A_1 v_1 = \rho_2 A_2 v_2. \]

Note:

\[
\frac{\Delta m}{\Delta t} = \frac{\rho \Delta V}{\Delta t} = \frac{\rho A_1 \Delta t}{\Delta t} = \rho A v
\]

13-8 Fluids in Motion; Flow Rate and the Equation of Continuity

If the density doesn’t change—typical for liquids—this simplifies to \[ A_1 v_1 = A_2 v_2. \] Where the pipe is wider, the flow is slower.

In humans, blood flows from the heart into the aorta, from which it passes into the major arteries. These branch into the small arteries (arterioles), which in turn branch into myriads of tiny capillaries. The blood returns to the heart via the veins. The radius of the aorta is about 1.2 cm, and the blood passing through it has a speed of about 40 cm/s. A typical capillary has a radius of about 4 x 10⁻⁴ cm, and blood flows through it at a speed of about 5 x 10⁻⁴ m/s. Estimate the number of capillaries that are in the body.

\[ r_A = 1.2 \text{ cm} = 1.2 \times 10^{-2} \text{ m} \]
\[ A_A = \pi r_A^2 \]
\[ v_A = 40 \text{ cm/s} = 0.4 \text{ m/s} \]
\[ r_c = 4 \times 10^{-4} \text{ cm} = 4 \times 10^{-6} \text{ m} \]
\[ A_c = \pi r_c^2 \]

Total area of capillaries = NA_c
\[ v_c = 5 \times 10^{-4} \text{ m/s} \]
\[ A_A v_A = N A_c v_c \]
\[ N = \frac{A_A v_A}{A_c v_c} = \frac{\pi r_A^2 v_A}{\pi r_c^2 v_c} = \frac{v_A}{v_c} \left( \frac{r_A}{r_c} \right)^2 \]
\[ = \frac{0.4}{5 \times 10^{-4}} \left( \frac{1.2 \times 10^{-2}}{4 \times 10^{-6}} \right)^2 \approx 7 \times 10^9 \]
Example 13-14: Heating duct to a room.

What area must a heating duct have if air moving 3.0 m/s along it can replenish the air every 15 minutes in a room of volume 300 m$^3$? Assume the air’s density remains constant.

Treat the room as a wider duct

$$A_1 v_1 = A_2 v_2 = A_2 \frac{\ell_2}{t_2} = \frac{V_2}{t_2}$$

$$A_1 = \frac{V_2}{v_1 t_2} = \frac{300}{3.0 \times 15 \times 60} = 0.11 \text{ m}^2$$

Problems:

1. Convert a flow rate of 150 L/min into m$^3$/s volume flow rate.

Solution:

1 L is the volume of a cube with sides of 100 mm
1 mm = 10$^{-3}$ m

∴ 1 L = (100 mm)$^3$ = (100 x 10$^{-3}$ m)$^3$ = (10$^{-1}$ m)$^3$ = 10$^{-3}$ m$^3$

∴ 150 \( \frac{L}{\text{min}} \) = 150x10$^{-3}$ \( \frac{\text{m}^3}{\text{s}} \) = 2.5x10$^{-3}$ m$^3$/s
2. A pipe changes from a diameter of 500 mm to 250 mm. If the flow rate in the 500 mm section is 1 m$^3$/s:
   (a) What is the flow rate in the narrower section?
   (b) What is the flow velocity in both sections of the pipe?

Solution:

(a) Equation of Continuity $\rightarrow Av = \text{constant}$ so the flow rate is still 1 m$^3$/s

(b) Larger section: $r = 0.25$ m
   $\rightarrow A = \pi r^2 = \pi \times 0.25^2 = 0.1963$ m$^2$
   Flow Rate: $Q = Av \rightarrow v = Q/A = 1/0.1963 = 5.09$ m/s

Smaller section: $r = 0.125$ m
   $\rightarrow A = \pi r^2 = \pi \times 0.125^2 = 0.0491$ m$^2$
   Flow Rate: $Q = Av \rightarrow v = Q/A = 1/0.0491 = 20.37$ m/s

Additional note on Question 2

As $A_1v_1 = A_2v_2 \rightarrow \pi r_1^2v_1 = \pi r_2^2v_2 \Rightarrow \frac{v_2}{v_1} = \left(\frac{r_1}{r_2}\right)^2$

If the diameter (radius) of the tube is halved

$\rightarrow \frac{r_1}{r_2} = 2$ and $\frac{v_2}{v_1} = 4$

i.e. the flow velocity in the smaller section will be four times the velocity in the larger section if the diameter is halved
Bernoulli’s principle:

Where the velocity of a fluid is high, the pressure is low, and where the velocity is low, the pressure is high.

This makes sense, as a force is required to accelerate the fluid to a higher velocity.

Consider the work it takes to move a small volume of fluid from one point to another while its flow is laminar. Work must be done to accelerate the fluid, and also to increase its height. Conservation of energy gives Bernoulli’s equation:

\[ \frac{1}{2} \rho v_2^2 - \frac{1}{2} \rho v_1^2 = P_1 - P_2 - \rho g y_2 + \rho g y_1, \]

\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2. \]
Consider a volume ($\Delta V$) of fluid moving from position 1 to position 2 in time $\Delta t$

The mass of the fluid is $\Delta m = \rho \Delta V = \rho A_1 \Delta x_1 = \rho A_2 \Delta x_2$

Using **Conservation of Energy**

– The work done ($\Delta W$) by the forces $F_1$ and $F_2$ on the mass $\Delta m$ will equal the increase in total energy $E_K + E_P$

Note that the work done by $F_1$ is **positive** and the work done by $F_2$ is **negative** as it opposes the motion

Work done on $\Delta m$:

$$\Delta W = W_1 - W_2 = F_1 \Delta x_1 - F_2 \Delta x_2$$

(Work = Force x Distance)

$$\therefore \Delta W = F_1 v_1 \Delta t - F_2 v_2 \Delta t$$

(distance = velocity x time)

$$= P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t$$

But $Av =$ volume flow rate $= \Delta V / \Delta t$

$$\therefore \Delta W = P_1 \frac{\Delta V}{\Delta t} \Delta t - P_2 \frac{\Delta V}{\Delta t} \Delta t = P_1 \Delta V - P_2 \Delta V = (P_1 - P_2) \Delta V$$
Change in Kinetic Energy:

\[ \Delta E_K = \frac{1}{2} \Delta m v_2^2 - \frac{1}{2} \Delta m v_1^2 = \frac{1}{2} \Delta m (v_2^2 - v_1^2) = \frac{1}{2} \rho \Delta v (v_2^2 - v_1^2) \]

Change in Potential Energy:

\[ \Delta E_p = \Delta m g y_2 - \Delta m g y_1 = \Delta m g (y_2 - y_1) = \rho \Delta V g (y_2 - y_1) \]

So \[ \Delta W = \Delta E_K + \Delta E_p \]

\[ \therefore (P_1 - P_2) \Delta V = \frac{1}{2} \rho \Delta V (v_2^2 - v_1^2) + \rho \Delta V g (y_2 - y_1) \]

Cancelling \( \Delta V \) and rearranging gives:

**Bernoulli’s Equation**

\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \]

or \[ P + \frac{1}{2} \rho v^2 + \rho g y = \text{constant} \]

If there is no change in height \( (y = 0) \), then

\[ P + \frac{1}{2} \rho v^2 = \text{constant} \]

This shows that a fluid has its maximum pressure when it is stationary \( (v = 0) \) and as the flow velocity increases the pressure decreases.

Using Bernoulli’s Principle we can explain some common phenomena.
The ball is in a moving air stream where pressure is lower than atmospheric pressure. If the ball moves slightly out of the moving air stream it will be moving to a region of higher pressure and the pressure forces will push the ball back into the stream – this is a stable equilibrium. Note that the vertical component of the upwards force due to air pressure supports the weight of the ball.

**Demonstration of Bernoulli’s Principle**

(Click image to show video)

13-9 Bernoulli’s Equation

Example 13-15: Flow and pressure in a hot-water heating system.

Water circulates throughout a house in a hot-water heating system. If the water is pumped at a speed of 0.5 m/s through a 4.0-cm-diameter pipe in the basement under a pressure of 3.0 atm, what will be the flow speed and pressure in a 2.6-cm-diameter pipe on the second floor 5.0 m above? Assume the pipes do not divide into branches.
Example 13-15: Flow and pressure in a hot-water heating system.

\[ \rho_{H_2O} = 1000 \text{ kg/m}^3 \quad P_{\text{atm}} = 1.013 \times 10^5 \text{ Pa} \approx 10^5 \text{ Pa} \]

**Basement:** \( v_1 = 0.5 \text{ m/s} \)

\[ A_1 = \pi r_1^2 \quad (r_1 = 2.0 \text{ cm}) \]

\[ P_1 = 3.0 \text{ atm} = 3.0 \times 10^5 \text{ Pa} \]

\( y_1 = 0 \)

**2nd Floor:** \( v_2 = ? \)

\[ A_2 = \pi r_2^2 \quad (r_2 = 1.3 \text{ cm}) \]

\( P_2 = ? \)

\( y_2 = 5.0 \text{ m} \)

Equation of Continuity

\[ \frac{A_1 v_1}{v_2} = \frac{A_2}{A_2} \quad v_2 = \frac{A_2}{A_1} v_1 \left( \frac{r_1}{r_2} \right)^2 \]

\[ = 0.5 \left( \frac{2.0}{1.3} \right)^2 \approx 1.18 \text{ m/s} \]

---

Example 13-15: Flow and pressure in a hot-water heating system.

**Bernoulli's Equation:**

\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \]

\[ P_2 = P_1 + \frac{1}{2} \rho (v_1^2 - v_2^2) + \rho g (y_1 - y_2) \]

\[ = 3.0 \times 10^5 + \frac{1000(0.5^2 - 1.18^2)}{2} + 1000 \times 9.8 (0 - 5.0) \]

\[ = 300000 - 571 - 49000 = 250000 \text{ Pa} \]

\[ = \frac{2.5 \times 10^5}{10^3} = 2.5 \text{ atm} \]
Using Bernoulli’s principle, we find that the speed of fluid coming from a spigot on an open tank is:

\[ \frac{1}{2} \rho v_1^2 + \rho g y_1 = \rho g y_2 \]

or

\[ v_1 = \sqrt{2g(y_2 - y_1)}. \]

This is called Torricelli’s theorem.

**13-10 Applications of Bernoulli’s Principle: Torricelli, Airplanes, Baseballs, TIA**

Lift on an airplane wing is due to the different air speeds and pressures on the two surfaces of the wing.
13-10 Applications of Bernoulli’s Principle: Torricelli, Airplanes, Baseballs, TIA

A sailboat can move against the wind, using the pressure differences on each side of the sail, and using the keel to keep from going sideways.

13-10 Applications of Bernoulli’s Principle: Torricelli, Airplanes, Baseballs, TIA

A ball’s path will curve due to its spin, which results in the air speeds on the two sides of the ball not being equal; therefore there is a pressure difference.
E.g. 1 – Curving of spinning ball in flight.

Diagrams show the air flow relative to the ball
(a) shows streamlines moving around the non-spinning ball
As the streamlines are compressed the air velocity increases and the pressure decreases, however the reduction in pressure is the same on both sides of the ball and there is no net force due to pressure

(b) shows the situation when the ball is spinning
The spinning ball drags air in contact with its surface, increasing the air velocity at the top of the diagram and thus reducing the pressure.
At the bottom the air velocity will be reduced with a corresponding increase in pressure.
This difference in pressure causes a net deflection force as shown and the ball curves in flight.
A person with constricted arteries may experience a temporary lack of blood to the brain (TIA) as blood speeds up to get past the constriction, thereby reducing the pressure.

A venturi meter can be used to measure fluid flow by measuring pressure differences.
Bernoulli’s principle also applies to any case of a spinning ball e.g. cricket ball, golf ball tennis ball

Explain what happens if you hit a tennis ball with either top spin or back spin.

Can you explain the following?

Low Pressure System

Image courtesy the SeaWiFS Project, NASA/Goddard Space Flight Center, and ORBIMAGE
URL:http://earthobservatory.nasa.gov/NaturalHazards/natural_hazards_v2.php3?img_id=2108
Problems:

1. A severe low pressure system is known as a **cyclone**. If the fastest winds in a cyclone are travelling at 200 km/h, what must be the pressure difference between that air and stationary air at 101.3 kPa?

(The density of air is 1.29 kg/m³)

For simplicity assume that the density of air is constant and that there is no variation in height.

Solution:

Convert 200 km/hr into m/s

\[
\text{1 km} = 1000 \text{ m} \\
\text{1 hr} = 60 \text{ min} = 60 \times 60 = 3600 \text{ s}
\]

\[
\frac{200 \text{ km}}{\text{hr}} = \frac{200 \times 1000 \text{ m}}{3600 \text{ s}} = 55.6 \text{ m/s}
\]

Using Bernoulli’s Equation:

\[
P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2
\]

The difference in pressure is \((P_1 - P_2)\) if we consider \(v_1 = 0\) to be stationary air and \(v_2 = 55.6 \text{ m/s}\) to be the fast moving air.

Also assuming \(y_1 = y_2\) gives:
\( P_1 - P_2 = \frac{1}{2} \rho (v_2^2 - v_1^2) = \frac{1}{2} \times 1.29 \times (55.6^2 - 0) = 1994 \text{ Pa} = 1.99 \text{ kPa} \)

Note normal atmospheric pressure is = 101.3 kPa

1.99 kPa represents a drop in pressure of about 1.8%

Pressure where the air velocity is 200 km/h is about 995 millibar (standard pressure = 1013 mb)
2. In order to maintain water pressure, some houses store tap water in a tank in the ceiling. If a water tank is in the ceiling of a two storey house (9.0 m from ground level), what is the pressure of the water at ground level if the volume flow rate is 1.5 litres per minute? The pipes in the house can be assumed to have a constant cross-section.

Solution:
Using Bernoulli’s equation:
\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \]

Since the flow rate is constant we can remove the velocity term from the equation. This gives:
\[ P_2 - P_1 = \rho g (y_1 - y_2) = 1000 \times 9.8 (9 - 0) = 88200 \text{ Pa} = \textbf{88.2 kPa} \]

3. A circular water tank (diameter 2.00 m) has a small circular pipe near the base with a diameter of 15 mm.

(a) If the water tank contains fresh water with a density of 998 kg/m³ what is the initial flow rate of water out of the pipe?

(b) When the water level has dropped to 1.00 m, what is the flow rate then?
Solution:

(a) Equation of Continuity
→ Flow Rate at Top of Tank = Flow Rate at Outlet
\[ A_1v_1 = A_2v_2 \]  
\( v_2 = \frac{A_1v_1}{A_2} = \frac{2^2}{0.015^2} \cdot v_1 = 17778 \cdot v_1 \)

Bernoulli’s Equation:
\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho gy_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho gy_2 \]
Pressure difference is very small so we can assume \( P_1 = P_2 \)
Also \( v_1 \ll v_2 \rightarrow v_1^2 \ll v_2^2 \) → can ignore \( v_1 \) term
Relative to the outlet: \( y_1 = 2 \) m and \( y_2 = 0 \)

\[ \therefore \text{Bernoulli's Equation reduces to: } \rho gy_1 = \frac{1}{2} \rho v_2^2 \rightarrow v_2 = \sqrt{2gy_1} = \sqrt{2 \times 9.8 \times 2} = 6.3 \text{ m/s} \]

(b) When \( y_1 = 1 \) m:

\[ \therefore \text{Bernoulli's Equation reduces to: } v_2 = \sqrt{2gy_1} = \sqrt{2 \times 9.8 \times 1} = 4.4 \text{ m/s} \]
Further Examples in Fluid Flow

Aerodynamic Lift – Airflow over wing

A small plane has a mass of 300 kg and a wing surface area of 30 m². What is the pressure difference required to produce enough lift to enable the aeroplane to fly? Note that the pressure is lower on top of the wing as the air velocity is higher (streamlines are closer together).

You do not need to use Bernoulli’s Equation to calculate the pressures, rather you should work out the weight of the aircraft and the pressure difference required to lift that weight.

Weight of aircraft: \[ W = mg = 300 \times 9.8 = 2940 \text{ N} \]

Pressure difference: \[ \Delta P = \frac{F}{A} = 2940/30 = 98 \text{ Pa} \]
In this example assume the plane is cruising at 180 km/h
= (180 x 1000)/(60 x 60) = 50 m/s.

The air velocity below the wing will be 180 km/h and the density
of the air is 1.29 kg/m$^3$. What is the air velocity above the wing
surface to provide the necessary pressure difference of 98 Pa?

Using Bernoulli’s Equation:

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho gy_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho gy_2$$

and noting that the height difference is negligible ($y_1 = y_2$), then
Bernoulli’s Equation gives:

$$v_2 = \sqrt{\frac{2(P_1 - P_2) + \rho v_1^2}{\rho}} = \sqrt{\frac{2(98)+1.29\times50^2}{1.29}}$$

$$= 51.5 \text{ m/s} = 51.5 \times 10^{-3} \times 60 \times 60 = 185 \text{ km/hr}$$

---

**Stagnation Pressure**

If we follow a streamline that ends on the nose of an aircraft we
discover the stagnation pressure – the maximum pressure
obtainable along a streamline – given by $P + \frac{1}{2} \rho v^2$

This is the point where all the kinetic energy (per unit volume) is
converted to pressure increase
Dynamic Pressure

In Bernoulli’s equation $\frac{1}{2}\rho v^2$ is often referred to as the **dynamic pressure**.

As demonstrated in the previous questions, the dynamic pressure dominates lift in most circumstances (elevation effects are small enough to be neglected).

**The lift force is ~ the dynamic pressure x the planform area** (area of wing viewed from above)

Some examples of lift force per unit area:
- 72 N/m² for the Wright Brother’s plane
- 7200 N/m² for a Boeing 747
- 50 N/m² for a bee

---

Venturi Flow Meter

The principle of the venturi flow meter can be understood from the above diagram. Using Bernoulli’s Equation (with $y_1 = y_2$) and the Equation of Continuity gives:

$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$  and since $A_1v_1 = A_2v_2 \rightarrow v_2 = \frac{A_1}{A_2} \cdot v_1$

$$\Delta P = P_1 - P_2 = \frac{1}{2}\rho v_1^2 \left[\frac{A_1^2}{A_2^2} - 1\right] \rightarrow v_1 = \frac{\sqrt{2\Delta P}}{\rho \left[\frac{A_1}{A_2}\right]^2 - 1}$$

$\Delta P$ can be calculated from: $\Delta P = \rho gh$
A more practical version is shown at right

Problem:
A venturi flow-meter like the one shown above is used to measure the fluid velocity of air (density 1.29 kg/m$^3$) as it passes around an aircraft. What is the velocity of the aircraft given that:

The U-Tube is filled with mercury ($\rho_{\text{Hg}} = 13.6 \times 10^3$ kg/m$^3$)
$\Delta h = 350$ mm.
The cross-sectional areas of the flow-meter are 10.0 cm$^2$ and 3.00 cm$^2$ respectively.

Solution:
The pressure difference is: $\Delta P = \rho_{\text{Hg}} g \Delta h$

$$\Delta P = 13.6 \times 10^3 \times 9.8 \times 350 \times 10^{-3} = 46.6 \times 10^3 \text{ Pa} = 14.6 \text{ kPa}$$

$$v_1 = \sqrt{\frac{2 \Delta P}{\rho \left(\frac{A_1}{A_2}\right)^2 - 1}} = \sqrt{\frac{2 \times 46.6 \times 10^3}{1.29 \left(\frac{10}{3}\right)^2 - 1}}$$

$$= 84.5 \text{ m/s} = 84.5 \times 10^{-3} \times 60 \times 60 = 304 \text{ km/h}$$

Note that in this case you can get away without converting areas to m$^2$ as they are only used as a ratio.
The Pitot tube is used as an air speed indicator on aircraft.

\[ \frac{v}{\sqrt{\frac{2 \Delta P}{\rho}}} \]

Sample Multichoice Questions – Pressure, Ideal Fluid Flow

1. Usually, atmospheric pressure
   (a) is equal to zero absolute pressure.
   (b) is equal to negative gauge pressure.
   (c) is equal to zero gauge pressure. **Correct Answer**
   (d) is equal to positive gauge pressure.

2. Which of the following pressures is the highest?
   (a) Atmospheric pressure on the earth’s surface at sea level.
   (b) Blood pressure in a major artery. **Correct Answer**
   (c) Atmospheric pressure on the moon’s surface.
   (d) Atmospheric pressure on top of Mount Everest.
   (e) Pressure in the pleural cavity.
3. A nurse exerts a force of 30 N onto the top surface of a syringe plunger. If the contact area is 2.0 cm$^2$, then the pressure on the plunger as exerted by the nurse is:

Note: 2 cm$^2$ is equal to 0.0002 m$^2$

(a) $30 / (0.00020)$ Pa.  
(b) $30 / (0.020)$ Pa.  
(c) $30 / (0.0002)$ N.  
(d) $30 / 2.0$ Pa.  
(e) $30 / 2.0$ newton per cm.

**Note (c) has the wrong units**

4. Aortal blood pressure has a typical value of 120 mmHg to 130 mmHg at systole. This value

(a) is a pressure above atmospheric pressure.  
(b) is a pressure less than atmospheric pressure.  
(c) is a negative gauge pressure.  
(d) is a static atmospheric gauge pressure.  
(e) can only be measured by use of a mercury based device.

5. A force of 120 N is applied to a surface of area 3 m$^2$. What is the pressure on the surface.

(a) 40 Pa  
(b) 60 Pa  
(c) $3 / 120$ Pa  
(d) 360 Pa  
(e) $3 \times 120 \times 10$ Pa

6. The static pressure increases with depth in a fluid because:

(a) the density of fluid increases with depth  
(b) of the weight of the fluid above  
(c) of the difference in the buoyancy force  
(d) the viscosity of fluid becomes larger  
(e) the increased diffusion produces a greater force.
7. Blood flow speed in the capillaries is less than in the aorta because:
   (a) the narrow capillaries have a higher flow resistance.
   (b) the narrow capillaries exert a higher friction force on the blood.
   (c) the aorta is closer to the pulsing heart.
   (d) the total cross section area of ALL capillaries is greater than the cross section area of the aorta.
   (e) the capillary has a smaller cross-sectional area.

8. Water moves along a tube which increases in cross-sectional area. As the water moves from narrow to wider sections of the tube, then
   (a) increased friction between the water and tube wall will decrease the speed in the narrow section.
   (b) water (volume) flow rate will be less in the narrow section of the tube.
   (c) water speed will be greater in the narrow section.
   (d) water (volume) flow rate will be greater in the narrow section.
   (e) water speed will not change from narrow to wider section.

   **Equation of Continuity:** \[ A_1v_1 = A_2v_2 \]
9. Imagine holding two bricks under water. Brick A is just below the surface and Brick B is at a greater depth. The force required to hold brick B in place is:
   (a) Larger
   (b) The same as
   (c) Smaller
   than the force required to hold brick A in place.

\[ F_{\text{TOT}} = \text{weight} - F_B \quad (F_B \text{ does not depend on depth}) \]

10. When a hole is made in the side of a container holding water, water flows out and follows a parabolic trajectory. If the container is dropped in free fall, the water flow:
   (a) Diminishes
   (b) Stops altogether
   (c) Goes out in a straight line
   (d) Curves upwards

   There is no pressure difference in free fall

11. A container is filled with oil and fitted on both ends with pistons. The area of the left piston is 10 mm\(^2\) while the right is 10 000 mm\(^2\). What force must be exerted on the left piston to keep a 10 000 N car on the right at the same height.

\[
\frac{F_1}{A_1} = \frac{F_2}{A_2} \\
F_1 = F_2 \times \frac{A_1}{A_2} = \frac{10000 \times 10}{10000} = 10 \text{ N}
\]
12. A 200 tonne ship enters the lock of a canal. The fit between the sides of the ship and the lock is so tight, that less than 200 tonnes of water is left in the lock. Can the ship still float if the quantity of water left in the lock is much less than the ships weight?

(a) Yes, as long as the level of the water reaches the waterline of the ship

(b) No, the ship touches the bottom because it weighs more than the water in the lock

13. Two cups are filled to the same level with water. One of the two cups has two ice cubes in it. Which weighs more?

(a) The cup without ice cubes

(b) The glass with two ice cubes

(c) The two weigh the same

14. A lead weight is fastened on top of a large solid piece of styrofoam that floats in a container of water. Because of the weight of the lead, the water line is flush with the top surface of the styrofoam. If the piece of styrofoam is turned up-sidedown so the weight is suspended beneath it:

(a) The arrangement sinks

(b) The water line is below the top surface of the styrofoam

(c) The water line is still flush with the top surface

15. A siphon has been started, fluid is now flowing through it. What is the velocity at the top of the tube in relation to the velocity of the exiting fluid:

(a) The velocity is greater at the top

(b) The velocity is less (but not zero) at the top

(c) The velocity is zero at the top

(d) The velocity is the same at the bottom
Notes on question 14:

With the lead on top the buoyancy force is due to the weight of the water displaced by the styrofoam and this is equal to the total weight of the styrofoam + lead.

When the styrofoam is turned over (and initially held so that its surface is level with the water surface) the total weight is unchanged however the buoyancy force will increase due to the increased displacement due to the volume of the lead. There will therefore be a net upwards force as buoyancy > weight and the top of the styrofoam will rise above the water surface until equilibrium is attained.

In practice these equations are only an approximation as they are based on an 'ideal' fluid

Real fluids have internal frictional forces called **viscosity** and are not incompressible

For low viscosity fluids this is a good approximation for most liquids however gases are compressible

Bernoulli’s Equation gives qualitative results for gas flow; however the approximations are only good for low pressure differences, i.e. low flow velocities

In a real fluid the flow may be turbulent which further complicates the issue

Real flow meters often use a pressure gauge to measure pressure difference and need to be calibrated in order to obtain accurate results
Real Fluid Flow

So far we have only studied the properties of ‘ideal’ fluids. Additional forces come into play when ‘real’ fluids are in motion.

The most important of these is viscosity.

Viscosity is the internal friction in a fluid which tends to prevent it from flowing when subjected to an applied force.

High-viscosity fluids resist flow; low-viscosity fluids flow easily.

The tendency of a fluid to drag adjacent layers of fluid along with it means that within a tube the velocity is greatest at the centre and least (zero) at the tube wall (fluid molecules adhere to the wall).

Velocity profile in a tube

The flow can be considered in terms of concentric cylinders (a). A short time later the central cylinders have moved further (b).
13-11 Viscosity

Real fluids have some internal friction, called viscosity.

The viscosity can be measured; it is found from the relation

\[ F = \eta A \frac{v}{\ell} \]

where \( v/\ell \) is the velocity gradient.

An apparatus for measuring viscosity is represented below

The force \( F \) is proportional to the area \( A \) and the velocity gradient \( v/\ell \)

\[ F = \eta A \frac{v}{\ell} \]

The constant \( \eta \) (Greek letter “eta”) is called the viscosity of the liquid and is measured in pascal second (Pa.s).
### Table of Viscosities

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Temperature (°C)</th>
<th>η/10^-3 (Pa·s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>20</td>
<td>0.0182</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0193</td>
</tr>
<tr>
<td>Carbon Dioxide</td>
<td>20</td>
<td>0.0147</td>
</tr>
<tr>
<td>Helium</td>
<td>20</td>
<td>0.0196</td>
</tr>
<tr>
<td>Whole Blood</td>
<td>37</td>
<td>4.0</td>
</tr>
<tr>
<td>Glycerine</td>
<td>20</td>
<td>1490</td>
</tr>
<tr>
<td>Methanol</td>
<td>20</td>
<td>0.584</td>
</tr>
<tr>
<td>Castor Oil</td>
<td>20</td>
<td>986</td>
</tr>
<tr>
<td>Mercury</td>
<td>20</td>
<td>1550</td>
</tr>
<tr>
<td>Water</td>
<td>0</td>
<td>1.78</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.656</td>
</tr>
</tbody>
</table>

The listing of values as η/10^-3 Pa·s indicates that the viscosity of mercury is 1550 x 10^-3 Pa·s

---

**Problem:**

A metal plate of area 200 cm² = 0.02 m² moves at a constant speed of 100 mm/s = 0.1 m/s parallel to another much larger plate. The 0.1 mm = 10^-4 m space between the plates is filled with an oil of viscosity 0.15 Pa.s. What is the force driving the moving plate?

**Solution:**

\[
F = \eta A \frac{\Delta v}{\Delta y} = 0.15 \times 0.02 \times \frac{0.1}{10^{-4}} = 3 \text{ N}
\]
The rate of flow in a fluid in a round tube depends on the viscosity of the fluid, the pressure difference, and the dimensions of the tube.

The volume flow rate is proportional to the pressure difference, inversely proportional to the length of the tube and to the pressure difference, and proportional to the fourth power of the radius of the tube.

This has consequences for blood flow—if the radius of the artery is half what it should be, the pressure has to increase by a factor of 16 to keep the same flow.

Usually the heart cannot work that hard, but blood pressure goes up as it tries.
More on Flow in Tubes

Laminar Flow

The simplest type of flow is **laminar flow**

Laminar means “in layers”, and we imagine layers of fluid sliding smoothly over each other in laminar flow. Laminar flow is also called **streamline flow**

We know from experience that the flow of a fast flowing fluid often breaks up and becomes **turbulent**

For example, water flowing in a river can develop eddies (whirlpools) when flowing past a submerged tree trunk, instead of flowing in smooth curves
Flow in Tubes
Bernoulli’s equation shows that the pressure in a tube depends on the flow velocity and is greatest where the velocity is lowest. When viscosity is taken into account there is also a pressure drop along the tube due to energy losses. These situations are illustrated below:

(a) represents an ideal (non-viscous) fluid and pressure (as indicated by the rise of liquid in the manometer tubes) depends on fluid velocity
(b) represents the case of a viscous fluid showing pressure drop due to viscosity.

Poiseuille’s Equation
When a pressure difference causes fluid flow through a horizontal cylindrical tube, the flow rate is limited by the viscosity of the fluid.

If there was no viscosity, fluid would flow through a level tube without an applied force. Due to viscosity, a pressure difference is required for flow.

The pressure in a fluid at rest depends only on the depth. When the fluid is flowing the situation is more complex. Viscosity (η) causes the pressure to drop along the length of the tube. E.g. blood flow along an artery → the pressure drops along the length of the artery. For a tube of radius R, length L, with pressure difference ΔP = P₁ – P₂ across the ends, the flow rate (Q) is given by Poiseuille’s Equation:

\[ Q = \frac{\pi R^4 (P_1 - P_2)}{8\eta L} \]
**Poiseuille’s Law** could be applied to the case of atherosclerosis (build up of fatty acid deposits in arteries)

If the deposits reduce the effective radius (or diameter) of (for example) a coronary artery to \( \frac{1}{2} \) its normal size, then the flow of blood is reduced to

\[ \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16} \text{th of the normal flow rate} \]

In practice this will be partially compensated for by an increase in blood pressure to maintain sufficient flow through the artery.

---

**Problem:**

24. Engine oil (assume SAE 10) at 25°C with a viscosity of 0.1 Pa.s passes through a fine 1.80 mm diameter tube in a prototype engine. The tube is 5.5 cm long. What pressure difference is required to maintain a flow rate of 5.6 mL/min?

**Solution:**

\[
Q = \frac{\pi R^4 (P_1 - P_2)}{8\eta L}
\]

\[
\Delta P = P_1 - P_2 = \frac{8\eta LQ}{\pi R^4}
\]

\[
\eta = 0.1 \text{ Pa·s} \\
D = 1.8 \text{ mm} \rightarrow R = 0.9 \text{ mm} = 9 \times 10^{-4} \text{ m} \\
L = 55 \text{ mm} = 0.055 \text{ m} \\
Q = 5.6 \text{ mL/min} \rightarrow
\]

\[
\begin{align*}
5.6 \text{ mL} &= 5.6 \times 10^{-6} \text{ m}^3 \\
\text{min} &= 60 \text{ s} \\
\therefore Q &= 9.33 \times 10^{-8} \text{ m}^3/\text{s}
\end{align*}
\]

\[
\begin{align*}
\Delta P &= 8 \times 0.1 \times 0.055 \times 9.33 \times 10^{-8} \\
\pi \left(9 \times 10^{-4}\right)^4 &= 1992 \text{ Pa}
\end{align*}
\]
SAE Viscosity

SAE represents the Society of Automotive Engineers viscosity classification. It is the most widely used system to classify the viscosity of engine lubricants.

There are two series of SAE viscosity, those with the suffix “W” for winter conditions and those without the “W”. Multigrade oils can meet two different temperature specifications (e.g., 20W50) meets spec for ambient temperatures of 20 and 50°C.

Concept Tests:

1. Water flows from a large tank into a pipe and then into a bucket. The pipe is then replaced by another one with the same length but twice the diameter. The pressure difference between the two ends of the pipe is then:
   - (a) the same as previously
   - (b) doubled
   - (c) halved
   - (d) 1/16 of the original value
   - (e) 16 times the original value

\[ \Delta P \text{ depends only on } P_1 \text{ at bottom of tank and } P_2 = P_{\text{atm}} \text{ at outlet} \]
2. A piece of stainless steel tubing that delivers fluid at a constant flow rate is replaced with a tube that has three times the diameter. What length must the new tube be for the flow rate to remain the same if the pressure drop along the tube is the same?

(a) 3 times the length of the original piece  
(b) 9 times the length of the original piece  
(c) **81 times the length of the original piece**  
(d) less than the length of the original piece  
(e) none of the other answers

$$Q_1 = \frac{\pi R_1^4 \Delta P}{8\eta L_1} = Q_2 = \frac{\pi R_2^4 \Delta P}{8\eta L_2}$$

$$\frac{R_1^4}{L_1} = \frac{R_2^4}{L_2} \rightarrow L_2 = \left(\frac{R_2}{R_1}\right)^4 L_1 = 3^4 L_1 = 81 L_1$$

---

**Transition from Laminar to Turbulent Flow**  
**- Reynolds Number**

The Reynolds Number is a single number that depends on all the relevant flow parameters and can tell us approximately when flow instabilities such as eddies and turbulence will occur.

At low velocities the flow in a tube will be laminar. If the pressure difference ($\Delta P$) is increased the flow rate will increase, however at a critical point eddies are formed, and the flow becomes turbulent.

Once turbulence starts energy is lost to the eddies - the flow becomes noisy and much greater pressure differences are required to produce even a small increase in flow rate.

Reynolds number is often expressed in different forms depending on the situation being studied.
Flow Through Tubes
For flow through tubes Reynolds number is often expressed as

\[ N_R = \frac{\rho \bar{V} D}{\eta} \]

where: \( \rho \) is the density of the fluid,
\( \bar{V} \) is the average velocity of fluid flow,
\( D \) the diameter of the tube
\( \eta \) the viscosity of the fluid

In tubes it is found experimentally that:

- \( N_R < 2000 \) flow is laminar
- \( N_R > 3000 \) flow is turbulent
- \( 2000 < N_R < 3000 \) flow is unstable
  (may change from laminar to turbulent or vice versa)

Note that the Reynolds number is dimensionless (has no units) so the same value would be obtained in any consistent set of units

Problem:

The average speed of blood in the aorta (\( D = 20 \text{ mm} \)) during the resting part of the heart’s cycle is about 300 mm/s. Is the flow laminar or turbulent?

Solution:

\[ N_R = \frac{\rho \bar{V} D}{\eta} = \frac{1060 \times 0.3 \times 0.02}{4 \times 10^{-3}} = 1590 \]

Since \( N_R < 2000 \) flow would be laminar
The graph at left shows how the flow rate $Q$ varies with an increase in pressure across the ends of a tube. The horizontal axis is the pressure gradient $\Delta P/L$.

The critical velocity $v_c$ at which the flow may become turbulent is given by:

$$N_R \approx \frac{\rho v_c D}{\eta} \rightarrow v_c \approx \frac{N_x \eta}{\rho D} \approx \frac{2000 \eta}{\rho D}$$

Once turbulence sets in the flow resistance increases dramatically.

An example from physiology:

Curve I represents a normal artery while Curve II is the case for an artery partly constricted by atherosclerosis (build up of fatty acid deposits in arteries). Because of the presence of the disk-shaped cells in whole blood, the real flow behaviour is more complicated; but, to a first approximation the diagram is valid and will serve as the basis for a discussion on the mechanics of blood flow.
Flow rates: \( Q_a \rightarrow \text{Resting flow rate} \)
\( Q_b \rightarrow \text{Flow rate during exercise} \)

For the normal healthy person the blood flow is laminar at both \( Q_a \) and \( Q_b \).

For the person suffering atherosclerosis the blood flow is laminar at \( Q_a \) (resting) however the blood pressure required to maintain the flow rate is greater than in the normal person.

If the flow rate is increased to \( Q_b \) then the flow becomes turbulent and the heart must work even harder to maintain the required blood pressure.

Flow Around Objects:

In considering flow around objects the Reynolds number is usually expressed in the form:

\[
N_R = \frac{\rho L v}{\eta}
\]

where:
\( \rho \) = density
\( L \) = characteristic length
\( v \) = characteristic velocity
\( \eta \) = viscosity

For example, for flow past a post in a very wide channel, \( L \) would be given by the post diameter and \( v \) would be the flow velocity far from the post.
For low values of $N_R$, flow is essentially laminar.

Illustration at right: $N_R = 0.16$

Slight instability

Illustration at right: $N_R = 1.54$

Once $N_R$ exceeds about 20, small vortices appear behind the cylinder. When $N_R$ is about 100, these vortices (eddies) periodically peel off and move with the flowing fluid.

Note that for every “clockwise” spiral (eddy) there is an “anticlockwise” spiral (eddy). These are more correctly called a vortex and are remarkably stable entities.

For example, the shedding of vortices from the wings of a 747 can affect aircraft travelling several hundreds of metres behind.
When $N_R$ is larger still, the flow becomes chaotic (turbulent) behind the cylinder and eventually, a boundary layer between the turbulent region and the laminar region develops.

At still higher values of $N_R$, more periodicities and more complicated behaviour emerges. The true nature of turbulent flow remains poorly understood, even by the best physicists!

In everyday life we see turbulent flow (e.g. smoke rising, water falling) which are both effects of increasing $v$ rather than length $L$.

**Drag Forces:**

For low velocity flow ($N_R < 1$) the drag force is proportional to velocity → $F_{\text{drag}} \propto v$

For high velocity flow ($N_R > 1000$) the drag force is approximately proportional to $v^2$

**Stokes’ Law**

When a spherical particle moves through a fluid, a frictional force due to the viscosity of the fluid acts on the particle. Stokes showed that under certain conditions, the drag force on a spherical particle is given by:

$$F_{\text{Drag}} = 6\pi \eta a v$$

In this equation $F$ is a drag force on the particle, $\eta$ is the fluid viscosity, $a$ is the radius of the sphere and $v$ is the velocity of the particle (relative to the fluid).
Limitations of Stokes’ Law

This form of Stokes’ Law assumes that the fluid velocity is the same as the particle velocity at the surface of the particle – the ‘stick’ boundary condition.

Stokes law is only valid for spherical particles in a container much larger than the size of the sphere and for small velocities.

Stokes law can be used to measure the viscosity of a liquid by dropping a small sphere into the liquid. The sphere must reach terminal velocity.

Using Stokes’ Law to Measure Viscosity

Consider forces acting on a falling sphere as shown in diagram:

$$ F_w = \text{weight of sphere} $$

$$ F_B = \text{buoyancy force (Archimedes’ Principle)} $$

$$ F_{\text{Drag}} = \text{drag force (Stokes’ Law)} $$

Using Stokes’ Law:

$$ F_w = F_B + F_{\text{drag}} $$

Where:

$$ F_{\text{drag}} = 6\pi R \eta v $$

$$ \eta = \frac{R F_{\text{drag}}}{6\pi v} $$
Using Stokes’ Law to Measure Viscosity - 2

$F_W$ and $F_B$ are constant and $F_{\text{drag}}$ increases as the velocity increases until equilibrium is obtained when $F_W - F_B - F_{\text{drag}} = 0$

At equilibrium there is no acceleration and the sphere reaches terminal velocity $v_T$

Using Stokes’ Law to Measure Viscosity - 3

Weight: $F_W = mg = \rho_s V g$

Buoyancy: $F_B = \rho_l V g$

Stokes’ Law: $F_{\text{drag}} = 6\pi \eta a v$

Apply Newton’s Second Law: $F_W - F_B - F_{\text{drag}} = ma$

at terminal velocity: $v = v_T$ and $a = 0$

$	herefore \rho_s V g - \rho_l V g - 6\pi \eta a v_T = 0$  $\rightarrow$  $6\pi \eta a v_T = (\rho_s - \rho_l) V g$
Using Stokes’ Law to Measure Viscosity - 4

Note that \( V = \frac{4}{3} \pi a^3 \)

\[ 6 \pi \eta a v_T = \left( \rho_s - \rho_l \right) \times \frac{4 \pi a^3 g}{3} \]

\[ \rightarrow 9 \eta v_T = 2 \left( \rho_s - \rho_l \right) a^2 g \]

Viscosity:

\[ \eta = \frac{2 \left( \rho_s - \rho_l \right) a^2 g}{9 v_T} \]

Terminal Velocity:

\[ v_T = \frac{2 \left( \rho_s - \rho_l \right) a^2 g}{9 \eta} \]

Aerodynamic drag

When an object moves through air so quickly that the assumption of laminar flow is violated, the drag force is no longer directly proportional to the velocity.

At high enough velocities, the fluid flow becomes turbulent and the drag force is more nearly proportional to the square of the velocity.

In this case, the drag force is often written as

\[ D = \frac{1}{2} C \rho A v^2 \]

where:  
- \( A \) is the cross-sectional area of the body (perpendicular to the velocity vector)  
- \( \rho \) is the density of air  
- \( C \) is the drag coefficient
Drag Coefficient

Typical values of drag coefficient ($C$)

- Thin disk: 1.10
- Small truck: 0.70
- 1988 BMW 735i: 0.32

Problems:

1. A steel ball-bearing, 8.0 mm in diameter, is dropped into a cylinder of glycerine, viscosity 1.49 Pa.s. The densities of steel and glycerine are $7.8 \times 10^3$ kg/m$^3$ and $1.26 \times 10^3$ kg/m$^3$ respectively. What is the terminal velocity of the ball-bearing?

Solution:

\[
\nu_T = \frac{2(\rho_s - \rho_l)a^2g}{9\eta} \\
= \frac{2(7800 - 1260)(4 \times 10^{-3})^2 \times 9.8}{9 \times 1.49} = 0.153 \text{ m/s}
\]

Here, $\rho_s = 7800$ kg/m$^3$, $\rho_l = 1260$ kg/m$^3$, $a = 4$ mm = $4 \times 10^{-3}$ m, and $\eta = 1.49$ Pa·s.
2. A bottle of light corn-syrup is taken from a refrigerator (T = 5°C) and a glass marble of density 2.5 x 10^3 kg/m^3 is dropped into it. The marble takes 45 s to sink to the bottom.

The diameter of the marble is 1.57 cm, the depth of the liquid is 12.1 cm, and its density is 1.2 x 10^3 kg/m^3. Assuming that the average velocity of the marble is equal to its terminal velocity, what is the viscosity of the syrup at that temperature?

Solution:

\[
\eta = \frac{2(\rho_s - \rho_l)a^2 g}{9v_T}
\]

\[
= \frac{2(2500 - 1200)(7.85 \times 10^{-3})^2 \times 9.8}{9 \times 2.689 \times 10^{-3}}
\]

\[
= 64.9 \text{ Pa} \cdot \text{s}
\]

3. Using the same data from the previous problem: If the bottle is kept out of the fridge for several hours and then the experiment is repeated, the marble takes 5 s to fall through the liquid. What is the viscosity of the syrup at room temperature (T = 20°C)?

Solution:

You could do this the hard way by calculating \(v_T\) and substituting data into equation as in previous question.

The easy way is to realise that the terminal velocity has increased by a factor of 9 and hence the viscosity will be 1/9 of the previous answer. \(\eta = \frac{69.4}{9} = 7.2 \text{ Pa} \cdot \text{s}\)
Summary of Chapter 13

- Phases of matter: solid, liquid, gas
- Liquids and gases are called fluids.
- Density is mass per unit volume.
- Specific gravity is the ratio of the density of the material to that of water.
- Pressure is force per unit area.
- Pressure at a depth \( h \) is \( \rho gh \).
- External pressure applied to a confined fluid is transmitted throughout the fluid.

Summary of Chapter 13

- Atmospheric pressure is measured with a barometer.
- Gauge pressure is the total pressure minus the atmospheric pressure.
- An object submerged partly or wholly in a fluid is buoyed up by a force equal to the weight of the fluid it displaces.
- Fluid flow can be laminar or turbulent.
- The product of the cross-sectional area and the speed is constant for horizontal flow.
Summary of Chapter 13

• Where the velocity of a fluid is high, the pressure is low, and vice versa.

• Viscosity is an internal frictional force within fluids.

• Additional Topics:
  • Transition from laminar to turbulent flow:
  • Reynolds Number
  • Stokes’ Law